

On an extension of Galligo's theorem concerning the Borel-fixed points on the Hilbert scheme.

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Abstract

Given an ideal I and a weight vector w which partially orders monomials we can consider the initial ideal $\text{in}_w(I)$ which has the same Hilbert function. A well known construction carries this out via a one-parameter subgroup of a GL_{n+1} which can then be viewed as a curve on the corresponding Hilbert scheme. Galligo [Gal79] proved that if I is in generic coordinates, and if w induces a monomial order up to a large enough degree, then $\text{in}_w(I)$ is fixed by the action of the Borel subgroup of upper-triangular matrices. We prove that the direction the path approaches this Borel-fixed point on the Hilbert scheme is also Borel-fixed.

1 Introduction

The purpose of this paper is to prove a first order infinitesimal version of a theorem of Galligo [Gal79]. Galligo's theorem states that in generic coordinates the initial ideal of any ideal is fixed by the action of the Borel subgroup of invertible upper-triangular matrices. This theorem has important consequences when translated to the Hilbert scheme. For example it immediately follows that any component, and any intersection of components on the Hilbert scheme will contain a Borel-fixed *point*. This follows since once we associate an ideal with its corresponding point on the Hilbert scheme, taking an initial ideal corresponds (in a way we make precise below) to the closure of a path parametrized by an appropriate one-parameter subgroup of a GL_{n+1} .

The infinitesimal version proven here says that not only is the limit point of this path Borel-fixed (Galligo's theorem translated to the Hilbert scheme), but also the path picks out a vector in the tangent space of the limit point which spans a subspace which is itself Borel-fixed. (Note that since the limit point is Borel-fixed, the action of the Borel group will descend to an action on the tangent space.)

This problem was posed to me by my PhD advisor David Bayer at Columbia University. I am grateful to him for many helpful conversations. The problem is also part of an on-going project to understand the local structure of the Hilbert scheme at a Borel-fixed point.

This paper is organized as follows: In sections 2 and 3 we quickly reproduce the relevant information needed about Hilbert schemes and Borel-fixed ideals. In section 4 we introduce a poset designed to capture combinatorially all the information of a Borel-fixed ideal. We discuss the poset and some of its properties briefly. The author believes the poset is in some ways the proper way to think about Borel-fixed ideals. Indeed, using the language of posets significantly eases statements of the later theorems. In section 5 we develop the notation used for

the tangent space to the Hilbert scheme at a Borel-fixed point. Sections 6 and 7 classify all the vectors of the tangent space which are Borel-eigenvectors (span a Borel-fixed subspace). Finally, in section 7 we prove the main result of this paper.

2 The Hilbert scheme

Throughout this paper we will work over an algebraically closed field K of characteristic 0. Let $\mathcal{H}_{\mathbf{p}(z)}^{\mathbf{P}^n}$ (or simply \mathcal{H}) denote the Hilbert Scheme parametrizing all subschemes of \mathbf{P}^n with a fixed Hilbert Polynomial $\mathbf{p}(z)$. We set $S = K[x_0, \dots, x_n]$ to be the homogeneous coordinate ring for \mathbf{P}^n , and for $d \geq 0$ we denote by S_d the vector space of the homogeneous forms of degree d in S , so that $S = \bigoplus_{d \geq 0} S_d$. Similarly for any homogeneous ideal $I \subseteq S$ we denote by I_d the vector space of its d th graded piece. Furthermore $I_{\geq d}$ denotes the truncated ideal with all elements of degree less than d removed. If $f_1, \dots, f_r \in S$ we will write (f_1, \dots, f_r) for the ideal generated by the f_i s.

The group $\mathrm{GL}(n+1, K)$ acts on S by extending its action on $S_1 \cong K^{n+1}$. The action on S_1 is computed in matrix form by taking $\{x_0, \dots, x_n\}$ to be a basis. If $g = (a_{ij})$ then

$$g(x_i) = g \cdot x_i = a_{0i}x_0 + \dots + a_{ni}x_n.$$

For a simple example, if $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = k[x, y]$ (where we set $x = x_0, y = x_1$) then $g(x) = x$ while $g(y) = x + y$ and hence $g(xy) = x^2 + xy$. $\mathrm{GL}(n+1, K)$ then acts on the set of ideals of S as well. Furthermore, if I is an ideal of S then $g(I)$ defines a scheme projectively equivalent to that defined by I , so we get an action of $\mathrm{GL}(n+1, K)$ on \mathcal{H} , and in fact on each of its irreducible components.

Given a scheme $Z \subseteq \mathbf{P}^n$, there are many ideals which define it. Among all such ideals there is a unique maximal one which contains all others. It can be obtained by the global sections functor $\mathcal{I} \mapsto \bigoplus_{d \geq 0} H^0(\mathcal{I}(d))$ applied to the ideal sheaf \mathcal{I} of Z , or equivalently by taking the primary decomposition of any ideal defining Z and removing the component associated to (x_0, \dots, x_n) . This operation is called *saturation* and the result of applying it to the ideal I will be denoted as I^{sat} .

Any two ideals defining Z will agree in large enough degree. Thus if I is the saturated ideal defining Z and d is large enough then the d -th graded piece I_d determines Z : the ideal generated by I_d agrees with I in degrees d and above; saturating the result recovers I .

The question of how large d should be is answered in part by noting the *regularity* of I suffices. We briefly recall what this is (the notion of regularity is due to Castelnuovo and Mumford; for a more detailed account see [Mum66]). For any coherent sheaf \mathcal{F} and nonnegative integer m , we say \mathcal{F} is m -regular if $H^i(\mathcal{F}(m-i)) = 0$ for all $i > 0$. The regularity of \mathcal{F} is the least integer m for

which \mathcal{F} is m -regular. Castelnuovo proved that if \mathcal{F} is m -regular, then: (i) \mathcal{F} is j -regular for each $j \geq m$; (ii) $\mathcal{F}(m)$ is generated by global sections. The regularity can also be characterized in terms of a minimal free resolution. This has the benefit of allowing one to define regularity for any finitely generated module. Let

$$0 \leftarrow F \leftarrow \bigoplus_j S(-e_{0j}) \leftarrow \cdots \leftarrow \bigoplus_j S(-e_{nj}) \leftarrow 0$$

be a minimal graded free resolution of the finitely generated module F . Then the regularity of F is $\max\{e_{ij} - i\}$.

It is very convenient that there is a finite integer bounding the regularity of all saturated ideals of schemes with a fixed Hilbert polynomial [Got78]. The smallest such integer is known as the *Gotzmann number*.

Given a Hilbert polynomial $\mathbf{p}(z)$ the Gotzmann number can be readily computed. Write $\mathbf{p}(z)$ in the form

$$\mathbf{p}(z) = g(m_0, \dots, m_s; z) := \sum_{i=0}^s \binom{z+i}{i+1} - \binom{z+i-m_i}{i+1}. \quad (1)$$

(See [Mac27] for details.) The integers m_0, \dots, m_s satisfy $m_0 \geq m_1 \geq \dots \geq m_s$ and are unique if we require $m_s \neq 0$, in which case we also get $s \leq n$ (in fact s is the dimension of the scheme). The Gotzmann number can be read off as m_0 .

Fix a Hilbert polynomial $\mathbf{p}(z)$ and let m be the Gotzmann number. Since a scheme with Hilbert polynomial $\mathbf{p}(z)$ can be identified with the vector space of degree m forms in its saturated defining ideal we can make a set-theoretical identification

$$\mathcal{H} \cong \{I_m \mid I = I^{\text{sat}}, \mathbf{p}_{S/I}(z) = \mathbf{p}(z)\}.$$

Let $s = \dim S_m$ and $r = s - \mathbf{p}(m)$. Any I_m in the above set has dimension equal to r , and is a subspace of S_m . This gives a set-theoretical inclusion of the above set into $G(r, S_m)$, the Grassmanian of r -dimensional subspaces of S_m . Thus we have a set-theoretical inclusion of \mathcal{H} into $G(r, S_m)$. One only needs to verify that this inclusion identifies \mathcal{H} with a closed subscheme of $G(r, S_m)$ with the proper scheme structure. This is accomplished by using the equations arising from the condition

$$\begin{aligned} V \in \{I_m \mid I = I^{\text{sat}}, \text{Hilb}(S/I) = \mathbf{p}(z)\} \\ \Downarrow \\ \dim(\{\text{Ideal generated by } V\}_{m+1}) = \dim S_{m+1} - \mathbf{p}(m+1). \end{aligned}$$

The fact that these equations scheme-theoretically define the Hilbert scheme was conjectured by Bayer [Bay82] and proven by Haiman and Sturmfels [HS04].

Throughout the paper we will be viewing the Hilbert scheme in this way; the points will correspond with vector subspaces of the vector space S_m , for an appropriately chosen m .

3 Borel-fixed ideals

If $A = (a_0, \dots, a_n) \in \mathbf{N}^{n+1}$ is a vector of non-negative integers, and $\mathbf{x} = (x_0, \dots, x_n)$, then we use the notation \mathbf{x}^A for the monomial $x_0^{a_0} \cdots x_n^{a_n}$. We will refer to A as the exponent vector of \mathbf{x}^A .

A *monomial order* (or *term order*) is a total multiplicative order on the set of monomials such that 1 is the least monomial. If $S = K[x_0, \dots, x_n]$ we will assume throughout that any monomial order $>$ satisfies $x_0 > x_1 > \cdots > x_n$.

If I is a monomial ideal of $S = K[x_0, \dots, x_n]$ (that is the minimal non-zero generators of I are monomials; equivalently I is fixed by the action diagonal matrices in $\mathrm{GL}(n+1, K)$), we set $\mathcal{M}(I)$ to be the set of monomials lying in I , and $\mathcal{M}(I_d)$ the set of monomials lying in I_d (the degree d monomials of I). Also $\mathcal{G}(I)$ will denote the minimal generating set of monomials for I . For a monomial \mathbf{x}^A we set $\max(\mathbf{x}^A)$ (or simply $\max(A)$) to be the index of the last variable dividing \mathbf{x}^A . That is

$$\max(\mathbf{x}^A) = \max(A) := \max\{i \mid x_i \mid \mathbf{x}^A\}$$

We similarly define $\min(\mathbf{x}^A)$ (and $\min(A)$). Note that with this definition it makes sense to set $\max(1) = -\infty$, and $\min(1) = +\infty$. Finally, $\deg(\mathbf{x}^A)$ (or $\deg(A)$) denotes the degree of the monomial \mathbf{x}^A , and $\deg_i(\mathbf{x}^A)$ (or $\deg_i(A)$) denotes the degree to which the variable x_i appears in \mathbf{x}^A .

Recall the action of $\mathrm{GL}(n+1, K)$ on the set of ideals of $S = K[x_0, \dots, x_n]$. An ideal is said to be *Borel-fixed* if it is fixed by the action of the Borel subgroup of $\mathrm{GL}(n+1, K)$ consisting of upper triangular matrices. Such ideals are stable ideals (defined in the next section) in the sense of Eliahou and Kervaire [EK90] and are precisely the strongly stable ideals in the sense of Peeva and Stillman [PS05]. Their corresponding points on the Hilbert scheme are of significant geometrical importance by virtue of their fixed point status. Moreover these ideals can be easily classified.

Proposition 3.1. *The Borel-fixed ideals are the ideals I such that*

- (1) *I is a monomial ideal.*
- (2) *If $\mathbf{x}^A \in I$ is a monomial, and $x_j \mid \mathbf{x}^A$, then for $i < j$, $\frac{x_i}{x_j} \mathbf{x}^A \in I$*

See, for example, [Eis95] chapter 15. The saturation and the regularity of a Borel-fixed ideal are easy to determine:

Theorem 3.2. *Let I be a Borel-fixed ideal, and $\mathcal{G}(I)$ its set of minimal monomial generators. Then the saturation of the ideal is generated by $\mathcal{G}(I)|_{x_n=1}$, that is one deletes the variable x_n in each of the generators.*

Proof. A monomial \mathbf{x}^A is in I^{sat} iff there is a power k such that $x_i^k \mathbf{x}^A \in I$ for $i = 0, \dots, n$. Since I is Borel-fixed this is the case iff $x_n^k \mathbf{x}^A \in I$ (proposition 3.1). Thus for any monomial $\mathbf{x}^A \in S$, we find $\mathbf{x}^A \in I^{\mathrm{sat}}$ if and only if there is some monomial $\mathbf{x}^B \in \mathcal{G}(I)$ such that \mathbf{x}^B divides $\mathbf{x}^A x_n^k$ for $k \gg 0$. One sees that this is equivalent to \mathbf{x}^A being a multiple of $\mathbf{x}^B|_{x_n=1}$. \square

Theorem 3.3. *The regularity of a Borel-fixed ideal is the highest degree of its minimal monomial generators.*

Proof. See [Bay82]. □

For any ideal I and term order $>$, the *initial ideal* $\text{in}_> I$ is the monomial ideal generated by the largest monomials appearing in all polynomials of I . The initial ideal can be obtained as a 1-parameter flat deformation of I (see section 8). A theorem of Bayer and Stillman and [BS87] states that *in generic coordinates, the regularity of an ideal is equal to the regularity of its initial ideal in the reverse lexicographic order*. Thus theorem 3.3 takes on great significance in the problem of determining regularity.

4 The poset $\mathcal{P}(m, n)$

Proposition 3.1 endows a Borel-fixed ideal with a combinatorial structure. Let $\mathcal{P} = \mathcal{P}(m, n)$ be the poset on the set of monomials of degree m in $S = k[x_0, \dots, x_n]$ with the relation \geq_B generated by the covering relation \succ_B where

$$\mathbf{x}^A \succ_B \mathbf{x}^B \iff \exists i < n \text{ such that } \mathbf{x}^A = \frac{x_i}{x_{i+1}} \mathbf{x}^B$$

We note that *every monomial order $>$ satisfying $x_0 > x_1 > \dots > x_n$ is a refinement of this Borel (partial) order*. Similar posets are considered in [MR99] and [Sne99].

For any Borel-fixed ideal I , the set $\mathcal{M}(I_m)$ of monomials in I_m will constitute a *filter* of $\mathcal{P}(m, n)$ – that is a subset $\mathcal{F} \subseteq \mathcal{P}$ such that $\mathbf{x}^B \in \mathcal{F}$ and $\mathbf{x}^A \geq_B \mathbf{x}^B$ implies $\mathbf{x}^A \in \mathcal{F}$. Dually, the *standard monomials* of degree m for I (monomials in $S_m \setminus I_m$) constitute an *order ideal* of \mathcal{P} , that is a subset $\mathcal{R} \subseteq \mathcal{P}$ such that if $\mathbf{x}^B \in \mathcal{R}$ and $\mathbf{x}^A \leq_B \mathbf{x}^B$ then $\mathbf{x}^A \in \mathcal{R}$.

For example there are two Borel-fixed points on the Hilbert scheme of 3 points in the plane. They are described by the Borel-fixed ideals (x^2, xy, y^2) and (x, y^3) . The first one is defined in degree 2, the second in degree 3. Figure 1 shows these ideals, the first in both degrees 2 and 3, the second in degree 3. They are represented as filters in the posets, with the filter elements circled.

The following proposition shows $\mathcal{P}(m, n)$ is a well-known poset. Let \mathbf{k} denote the k -element chain on $\{1, 2, \dots, k\}$, and $J(X)$ the poset on the order-ideals of the poset X , and finally $\langle z_1, \dots, z_k \rangle$ the order-ideal generated by z_1, \dots, z_k .

Proposition 4.1. *We have*

$$\mathcal{P}(m, n) \cong J(\mathbf{m} \times \mathbf{n}).$$

In particular $\mathcal{P}(m, n)$ is a distributive lattice.

Proof. The isomorphism is given by

$$x_0^{a_0} x_1^{a_1} \dots x_n^{a_n} \longleftrightarrow \langle (a_0, n), (a_0 + a_1, n-1), \dots, (a_0 + a_1 + \dots + a_{n-1}, 1) \rangle$$

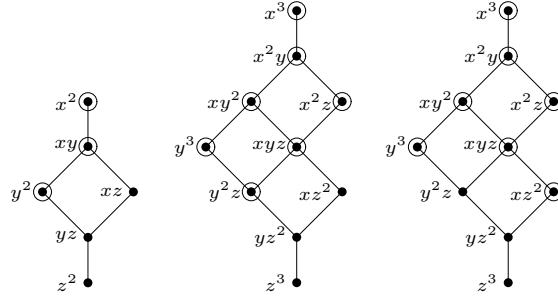


Figure 1: (x^2, xy, y^2) in degree 2, 3, and (x, y^3) in degree 3

where we omit (k, l) if $k = 0$. To see this is an isomorphism first note that any order-ideal R of $\mathbf{m} \times \mathbf{n}$ is uniquely described in the form $\langle (k_1, 1), (k_2, 2), \dots, (k_n, n) \rangle$ by taking k_i maximal such that $(k_i, i) \in R$, or setting $k_i = 0$ if no such pair is in R , and consider it as not occurring. Then note that $k_i \geq k_{i+1}$, for if this were not true then $(k_i + 1, i) \leq (k_{i+1}, i + 1)$ which would imply $(k_i + 1, i) \in R$, contradicting the maximality of k_i . Hence for the k_i s there are unique a_i s such that $k_i = a_0 + a_1 + \dots + a_{n-i-1}$. Finally the covering relations correspond: if $R = \langle (a_0, n), (a_0 + a_1, n - 1), \dots, (a_0 + a_1 + \dots + a_{n-1}, 1) \rangle$ then

$$\begin{array}{ccc}
 \mathcal{P}(m, n) & & J(\mathbf{m} \times \mathbf{n}) \\
 \\
 x_0^{a_0} \dots x_i^{a_i+1} x_{i+1}^{a_{i+1}-1} \dots x_n^{a_n} & \longleftrightarrow & R \cup \{(a_0 + \dots + a_i + 1, n - i)\} \\
 | & & | \\
 x_0^{a_0} \dots x_n^{a_n} & \longleftrightarrow & R
 \end{array}$$

That $\mathcal{P}(m, n)$ is a distributive lattice follows from the *fundamental theorem for finite distributive lattices*. See for example [Sta97], theorem 3.4.1. \square

Corollary 4.2.

$$\mathcal{P}(m, n) \cong \mathcal{P}(n, m)$$

Through the maps $\mathcal{P}(m, n) \rightarrow J(\mathbf{m} \times \mathbf{n}) \rightarrow J(\mathbf{n} \times \mathbf{m}) \rightarrow \mathcal{P}(n, m)$ one can construct the isomorphism $\mathcal{P}(m, n) \cong \mathcal{P}(n, m)$ explicitly. Let \mathbf{x}^A be a degree m monomial in the variables x_0, \dots, x_n . Write $\mathbf{x}^A = x_{\alpha_1} \dots x_{\alpha_m}$ where $\alpha_i \leq \alpha_{i+1}$ for $i = 0, \dots, m$. Set

$$b_i = \begin{cases} n - \alpha_m, & i = 0 \\ \alpha_{m-i+1} - \alpha_{m-i}, & i = 1, \dots, m-1 \\ \alpha_1, & i = m \end{cases}$$

If $B = (b_0, \dots, b_m)$ and $\mathbf{y} = (y_0, \dots, y_m)$ then the isomorphism $\mathcal{P}(m, n) \cong \mathcal{P}(n, m)$ identifies \mathbf{x}^A with \mathbf{y}^B . For example

$$x_0^2 x_1^3 x_3 = x_0 x_0 x_1 x_1 x_1 x_3 \rightarrow y_0^{3-3} y_1^{3-1} y_2^{1-1} y_3^{1-1} y_4^{1-0} y_5^{0-0} y_6^0 = y_1^2 y_4.$$

There are two much nicer ways to obtain the isomorphism. The first associates to a monomial its *bars and stars* representation. Then one flips the role of the bars with that of the stars and reads backwards. With the above example we find

$$x_0^2 x_1^3 x_3 \sim **|***||* \longrightarrow ||*|||**| \rightarrow |**|||*|| \sim y_1^2 y_4.$$

The second associates a monomial in $\mathcal{P}(m, n)$ with a path in a $m \times n$ grid from the southwest corner to the northeast corner which always moves either up or to the right in integral increments. Each unit rise in the path indicates a variable corresponding to the horizontal position. In our example the monomial $x_0^2 x_1^3 x_3$ is represented by the picture in figure 2.

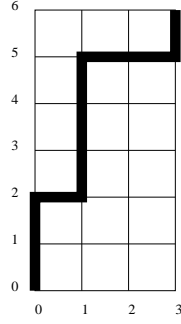


Figure 2: The grid representation of the monomial $x_0^2 x_1^3 x_3 \in K[x_0, \dots, x_3]$.

The image monomial is obtained by flipping the grid from southwest to northeast as in figure 3. Once again we obtain $y_1^2 y_4$.

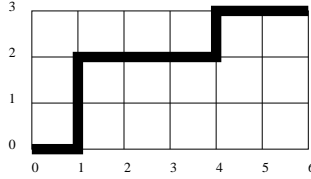


Figure 3: The grid representation of the monomial $y_1^2 y_4 \in K[y_0, \dots, y_6]$

Proposition 4.3. Let $\mathbf{x}^{A_1}, \mathbf{x}^{A_2}$ be monomials of degree m in $K[x_0, \dots, x_n]$, and set $\mathbf{y}^{B_1}, \mathbf{y}^{B_2}$ to be the corresponding monomials of degree n in $K[y_0, \dots, y_m]$. Then

$$\mathbf{x}^{A_1} <_{\text{Lex}} \mathbf{x}^{A_2} \iff \mathbf{y}^{B_1} <_{\text{RevLex}} \mathbf{y}^{B_2}.$$

Proof. Consider the *grid representations* of the monomials as described above. If $\mathbf{x}^{A_1} <_{\text{Lex}} \mathbf{x}^{A_2}$ then the first juncture at which the path corresponding to \mathbf{x}^{A_1} differs from that of \mathbf{x}^{A_2} must have the latter path going up, while the former going right. Hence after the flip the last juncture in which the paths for

\mathbf{y}^{B_1} and \mathbf{y}^{B_2} differ will have the latter path coming in from the left, while the former from underneath (see figure 4). One readily verifies this is equivalent to $\mathbf{y}^{B_1} <_{\text{RevLex}} \mathbf{y}^{B_2}$.

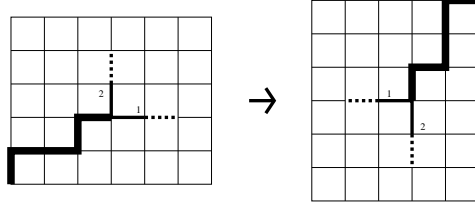


Figure 4: The effect of a flip on the ordering of two monomials.

Alternatively one can use the *bars and stars* representations and use the same logic. \square

We include here a lemma which we find demonstrates the interplay of the combinatorics of filters in $\mathcal{P}(m, n)$, and the algebra of their defining ideals.

Lemma 4.4. *Let I be a Borel-fixed ideal defined in degrees $\leq m$. Let $\mathcal{F} = I \cap \mathcal{P}(m, n)$ be the corresponding filter. If \mathbf{x}^A is a standard monomial of degree m ($\mathbf{x}^A \in S_m \setminus I_m$), then*

$$x_i \mathbf{x}^A \in I \iff \frac{x_i}{x_{\max(A)}} \mathbf{x}^A \in \mathcal{F}.$$

In particular, if \mathbf{x}^A is Borel maximal in $\mathcal{P}(m, n) \setminus \mathcal{F}$ (that is every greater monomial lies in \mathcal{F}) then

$$x_i \mathbf{x}^A \in I \iff i < \max(A).$$

Proof. Let $k = \max(A)$. If $x_i/x_k \mathbf{x}^A \in \mathcal{F}$ then certainly $x_i \mathbf{x}^A \in I$. Conversely suppose $x_i \mathbf{x}^A \in I$. Since I is generated in degrees less than or equal to m , there must be a monomial $\mathbf{x}^B \in \mathcal{F}$ and a variable x_j such that

$$x_i \mathbf{x}^A = x_j \mathbf{x}^B.$$

Certainly $i \neq j$ (since $\mathbf{x}^A \notin I$) and hence $j \leq \max(A) = k$. But then

$$\frac{x_i}{x_k} \mathbf{x}^A = \frac{x_j}{x_k} \mathbf{x}^B \in \mathcal{F}$$

since \mathcal{F} is a filter. \square

5 The tangent space to the Hilbert scheme at a Borel-fixed point.

Fix a projective space \mathbf{P}^n and a Hilbert polynomial $\mathbf{p}(z)$ and set \mathcal{H} to be the corresponding Hilbert scheme. If $z \in \mathcal{H}$ corresponds to the scheme $Z \in \mathbf{P}^n$

then it is known that

$$T_z \mathcal{H} = H^0 \mathcal{N}_{Z/\mathbf{P}^n}.$$

That is, the tangent space to $z \in \mathcal{H}$ is identified with the global sections of the normal sheaf to $Z \subset \mathbf{P}^n$.

Let z be a Borel-fixed point on \mathcal{H} , let the corresponding ideal sheaf be \mathcal{I} , and let m be the Gotzmann number for the corresponding Hilbert polynomial. The tangent space to this point will be a subspace of the tangent space to the Grassmanian $G(r, S_m)$. Recall what the tangent space to the Grassmanian looks like. Let $v \in G(r, S_m)$ correspond to the r -dimensional subspace V of S_m . Then

$$T_v G(r, S_m) = \text{Hom}(V, S_m/V).$$

We want to see how the tangent space to $z \in \mathcal{H}$ sits naturally as a subspace of $T_z G(r, S_m)$. Recall that $\mathcal{N}_{Z/\mathbf{P}^n} = \mathcal{H}om(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Z)$ so $T_z \mathcal{H} = H^0 \mathcal{N}_{Z/\mathbf{P}^n} = \text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Z)$. Now a map $\phi : \mathcal{I}/\mathcal{I}^2 \rightarrow \mathcal{O}_Z$ composes with the natural map $\mathcal{I} \rightarrow \mathcal{I}/\mathcal{I}^2$ to give $\tilde{\phi} : \mathcal{I} \rightarrow \mathcal{O}_Z$. Twisting by m we then get a map $\mathcal{I}(m) \rightarrow \mathcal{O}_Z(m)$. Then take global sections to get $H^0(\mathcal{I}(m)) \rightarrow H^0(\mathcal{O}_Z(m))$. Let $I = \bigoplus_{d \geq 0} H^0(\mathcal{I}(d))$. Since $m \geq \text{reg}(\mathcal{I})$, we see

$$H^0(\mathcal{I}(m)) = I_m \text{ and } H^0(\mathcal{O}_Z(m)) = S_m/I_m.$$

Hence we get the map $I_m \rightarrow S_m/I_m \in T_z G(r, S_m)$.

The tangent space to a point $z \in \mathcal{H}$ can also be identified with the space of first-order infinitesimal deformations of the corresponding scheme in \mathbf{P}^n . Specifically, if $z \in \mathcal{H}$ corresponds to the scheme defined by the ideal $I = I^{\text{sat}}$ then

$$\begin{aligned} T_z \mathcal{H} &\cong \{J \subset S[\varepsilon] \mid J \text{ flat over } K[\varepsilon], J|_{\varepsilon=0} = I\} \\ &\cong \{J_m \subset S_m[\varepsilon] \mid J_{\geq m} \text{ flat over } K[\varepsilon], J_{\geq m}|_{\varepsilon=0} = I_{\geq m}\} \end{aligned}$$

Now let $z \in \mathcal{H}$ be a Borel-fixed point whose corresponding scheme is defined by the Borel-fixed ideal $I = I^{\text{sat}}$. Set $\mathcal{F} = I \cap \mathcal{P}(m, n)$ to be the corresponding filter, and $\mathcal{R} = \mathcal{P}(m, n) \setminus \mathcal{F}$ the corresponding order-ideal of the standard monomials in degree m , as we defined in section 4. For notational reasons it is often simpler if we consider \mathcal{F} and \mathcal{R} as consisting of the exponent vectors of the monomials. In this respect we will switch back and forth between monomials and their exponent vectors and trust that no confusion will arise. Note that since the Borel subgroup fixes z it induces an action on $T_z \mathcal{H}$.

An arbitrary vector in the tangent space to z in $G(r, S_m)$ is given by a K -linear map $\phi : I_m \rightarrow S_m/I_m$ which can be described uniquely by

$$\phi(\mathbf{x}^A) = \sum_{B \in \mathcal{R}} c_{AB} \mathbf{x}^B, \quad A \in \mathcal{F}.$$

We denote this by the doubly-indexed vector

$$(c_{AB})_{A \in \mathcal{F}, B \in \mathcal{R}} = (c_{AB}) \in K^{\#\mathcal{F} \times \mathcal{R}} \cong T_z G(r, S_m).$$

This vector then lies in $T_z \mathcal{H}$ if and only if

$$J := \left(\mathbf{x}^A + \varepsilon \sum_{B \in \mathcal{R}} c_{AB} \mathbf{x}^B \mid A \in \mathcal{F} \right),$$

with $\varepsilon^2 = 0$, is flat over $K[\varepsilon]$ (that is J defines a first order infinitesimal deformation of I). We will set $\{e_{AB} \mid A \in \mathcal{F}, B \in \mathcal{R}\}$ to be the basis of $T_z G(r, S_m)$ where e_{AB} is the vector with 0s in every coordinate except the one with index (A, B) .

6 The maximal torus eigenvectors

In this section we classify those vectors of the tangent space to the Hilbert scheme at Borel-fixed point which are eigenvectors for the maximal torus subgroup of $\mathrm{GL}(n+1, K)$ consisting of diagonal matrices.

Lemma 6.1. *Let z be a Borel-fixed point on \mathcal{H} . Let m be the regularity of the sheaf of ideals \mathcal{I} of Z . Let I be the (Borel-fixed) ideal given by*

$$I_d = \begin{cases} H^0 \mathcal{I}(d) & \text{if } d \geq m \\ 0 & \text{otherwise} \end{cases}$$

Set \mathcal{F} to be the set of (exponent vectors of) monomials of I of degree m . Set \mathcal{R} to be the set of (exponent vectors of) the standard monomials of degree m . (Recall that \mathcal{F} is a filter of $\mathcal{P}(m, n)$, while \mathcal{R} is the complimentary order-ideal.) Then the infinitesimal deformation

$$\left(\mathbf{x}^A + \varepsilon \sum_{B \in \mathcal{R}} c_{AB} \mathbf{x}^B \mid A \in \mathcal{F} \right) = (c_{AB})_{A \in \mathcal{F}, B \in \mathcal{R}}$$

considered as an element of the tangent space to z is an eigenvector for the maximal torus of diagonal matrices if and only if there exists $K \in \mathbf{Z}^{n+1}$ of degree 0 such that

$$c_{AB} \neq 0 \implies B - A = K.$$

Proof. Let Λ be the diagonal matrix with diagonal entries $\lambda = (\lambda_0, \dots, \lambda_n)$. For the deformation $(c_{AB})_{A \in \mathcal{F}, B \in \mathcal{R}}$ set

$$r_A = \mathbf{x}^A + \varepsilon \sum_{B \in \mathcal{R}} c_{AB} \mathbf{x}^B.$$

Then

$$\Lambda \cdot r_A = \lambda^A \mathbf{x}^A + \varepsilon \sum_{B \in \mathcal{R}} c_{AB} \lambda^B \mathbf{x}^B$$

from which we get

$$\lambda^{-A} \Lambda \cdot r_A = \mathbf{x}^A + \varepsilon \sum_{B \in \mathcal{R}} c_{AB} \lambda^{B-A} \mathbf{x}^B$$

Hence

$$\Lambda \cdot (c_{AB}) = (\lambda^{B-A} c_{AB})$$

By varying the λ_i 's we see that the $B - A$ must be constant over those A, B for which $c_{AB} \neq 0$. Setting $K = B - A$ we get our result, and in fact

$$\Lambda \cdot (c_{AB}) = \lambda^K (c_{AB})$$

□

7 The Borel eigenvectors

For an eigenvector of the maximal torus as in lemma 6.1 set

$$\mathcal{F}' = \mathcal{F} \setminus \{A \in \mathcal{F} \mid \exists B \in \mathcal{R} \text{ such that } c_{AB} \neq 0\}$$

and

$$\mathcal{F}'' = \mathcal{F} \cup \{B \in \mathcal{R} \mid \exists A \in \mathcal{F} \text{ such that } c_{AB} \neq 0\}$$

We will say this eigenvector has *type* $(\mathcal{F}', \mathcal{F}'', K)$. Note that \mathcal{F}'' is determined in terms of \mathcal{F}' and K :

$$\mathcal{F}'' = \mathcal{F} \cup ((\mathcal{F} \setminus \mathcal{F}') + K) \quad (\text{disjoint union}).$$

So we may refer to this vector as having type (\mathcal{F}', K) .

If $(c_{AB})_{A \in \mathcal{F}, B \in \mathcal{R}}$ is an eigenvector for the maximal torus we can denote it by $(c_A)_{A \in \mathcal{F}}$ without confusion since for any $A \in \mathcal{F}$ there is at most one $B \in \mathcal{R}$ (namely $A + K$) such that $c_{AB} \neq 0$. Specifically, the notation $(c_A)_{A \in \mathcal{F}}$ (or even more simply (c_A)) for an eigenvector of type (\mathcal{F}', K) will refer to the ideal of $S[\varepsilon]$ generated by the elements

$$\mathbf{x}^A, \text{ for each } A \in \mathcal{F} \setminus \mathcal{F}', \quad \mathbf{x}^A + \varepsilon c_A \mathbf{x}^{A+K}, \text{ for each } A \in \mathcal{F}'.$$

The question of which vectors in $T_z \mathcal{H}$ are eigenvectors for the Borel group of upper-triangular matrices is a little more tricky. First observe that since the diagonal matrices are a subgroup of the Borel group we must have that such a vector is an eigenvector for the maximal torus.

Take an eigenvector for the maximal torus, of type $(\mathcal{F}', \mathcal{F}'', K)$. We say this vector is a *pseudo-eigenvector* for the Borel subgroup if its image under the action of any upper-triangular matrix also has type $(\mathcal{F}', \mathcal{F}'', K)$. Certainly an eigenvector for the Borel group is a pseudo-eigenvector.

Let $E_i \in \mathbf{Z}^{n+1}$, $i = 0, \dots, n$ be the vector with a 1 in the i 'th position, and 0's elsewhere (note that the '0 position' is the first coordinate). For $i = 1, \dots, n$ set $\Delta_i = E_{i-1} - E_i$. Notice that every covering relation in $\mathcal{P}(m, n)$ is of the form $A \prec_B A + \Delta_i$ for some i .

Lemma 7.1. *Let $(c_A)_{A \in \mathcal{F}}$ be an eigenvector for the maximal torus, of type $(\mathcal{F}', \mathcal{F}'', K)$. If (c_A) is a pseudo-eigenvector for the Borel subgroup then:*

(i) \mathcal{F}' and \mathcal{F}'' are both filters;

(ii) If $A, A + \Delta_i \in \mathcal{F} \setminus \mathcal{F}'$ then

$$c_{A+\Delta_i} = \frac{a_i}{b_i} c_A$$

where $a_i = \deg_i(A)$ and $b_i = \deg_i(A + K)$.

(iii) $(c_A)_{A \in \mathcal{F}}$ is an eigenvector for the Borel subgroup.

Proof. Let J be the corresponding ideal of $S[\varepsilon]$. Its generators are

$$\mathbf{x}^A, A \in \mathcal{F}', \quad \mathbf{x}^A + \varepsilon c_A \mathbf{x}^{A+K}, A \in \mathcal{F} \setminus \mathcal{F}'.$$

Let h be an arbitrary upper-triangular matrix. By assumption hJ has generators $\mathbf{x}^A + \varepsilon s_A \mathbf{x}^{A+K}$ where $s_A = 0 \iff A \in \mathcal{F}'$. Hence for $B \in \mathcal{F}$ there must be a relation

$$\mathbf{x}^B + \varepsilon s_B \mathbf{x}^{B+K} = \sum_{A \in \mathcal{F}'} (\lambda_A + \varepsilon \mu_A) h \cdot \mathbf{x}^A + \sum_{A \in \mathcal{F} \setminus \mathcal{F}'} (\lambda_A + \varepsilon \mu_A) h \cdot (\mathbf{x}^A + \varepsilon c_A \mathbf{x}^{A+K}).$$

This is equivalent to the two equations

$$\mathbf{x}^B = h \left(\sum_{A \in \mathcal{F}} \lambda_A \mathbf{x}^A \right) \quad (2)$$

and

$$s_B \mathbf{x}^{B+K} = h \left(\sum_{A \in \mathcal{F}} \mu_A \mathbf{x}^A + \sum_{A \in \mathcal{F} \setminus \mathcal{F}'} \lambda_A c_A \mathbf{x}^{A+K} \right) \quad (3)$$

If $B \in \mathcal{F}'$ then $s_B = 0$, and (3) becomes

$$0 = h \left(\sum_{A \in \mathcal{F}} \mu_A \mathbf{x}^A + \sum_{A \in \mathcal{F} \setminus \mathcal{F}'} \lambda_A c_A \mathbf{x}^{A+K} \right)$$

Since h is nonsingular and the sum on the right is over distinct monomials, we find $\mu_A = 0, A \in \mathcal{F}$ and $\lambda_A = 0, A \in \mathcal{F} \setminus \mathcal{F}'$. So (2) becomes

$$h^{-1} \mathbf{x}^B = \sum_{A \in \mathcal{F}'} \lambda_A \mathbf{x}^A.$$

Since $B \in \mathcal{F}'$ is arbitrary, as well as h , we see \mathcal{F}' is a filter.

On the other hand if $B \in \mathcal{F} \setminus \mathcal{F}'$ then $s_B \neq 0$ and (2) and (3) can be rewritten

$$h^{-1} \mathbf{x}^B = \sum_{A \in \mathcal{F}} \lambda_A \mathbf{x}^A$$

and

$$h^{-1}\mathbf{x}^{B+K} = \sum_{A \in \mathcal{F}} s_B^{-1} \mu_A \mathbf{x}^A + \sum_{A \in \mathcal{F} \setminus \mathcal{F}'} s_B^{-1} \lambda_A c_A \mathbf{x}^{A+K}$$

Since $\mathcal{F}'' = \mathcal{F} \dot{\cup} ((\mathcal{F} \setminus \mathcal{F}') + K)$ these show \mathcal{F}'' is a filter.

Lastly let $A \in \mathcal{F} \setminus \mathcal{F}'$. Set $a_i = \deg_i(A)$, and $b_i = \deg_i(A + K)$.

Let h_i be the upper-triangular matrix which sends $x_i \mapsto x_i + x_{i-1}$ and leaves the other variables fixed. We compute

$$h_i(\mathbf{x}^A + \varepsilon c_A \mathbf{x}^{A+K}) = \sum_{j=0}^{a_i} \binom{a_i}{j} \mathbf{x}^{A+j\Delta_i} + \sum_{j=0}^{b_i} \binom{b_i}{j} c_A \varepsilon \mathbf{x}^{A+K+j\Delta_i}$$

where $a_i = \deg_i(A)$, and $b_i = \deg_i(A + K)$. Set l to the maximal such that $A + K + l\Delta_i \in \mathcal{R}$. We already know that \mathcal{F}'' is a filter, and this implies that $l \leq a_i$. Then $h_i J$ contains

$$h_i(\mathbf{x}^A + \varepsilon c_A \mathbf{x}^{A+K}) = \sum_{j=l+1}^{a_i} \binom{a_i}{j} \underbrace{\mathbf{x}^{A+j\Delta_i}}_{\in h_i J} + \sum_{j=0}^l \binom{a_i}{j} \left[\mathbf{x}^{A+j\Delta_i} + \varepsilon c_A \frac{\binom{b_i}{j}}{\binom{a_i}{j}} \mathbf{x}^{A+K+j\Delta_i} \right]$$

Each individual term in the left-hand sum lies in $h_i J$ so the right-hand sum must be in $h_i J$. By assumption $h_i J$ has generators of the form

$$\{\mathbf{x}^B + \varepsilon s_B \mathbf{x}^{B+K} \mid B \in \mathcal{F}\}.$$

So $\mathbf{x}^{A+j\Delta_i} + \varepsilon s_{A+j\Delta_i} \mathbf{x}^{A+K+j\Delta_i} \in h_i J$. It follows that

$$\varepsilon \sum_{j=0}^l \binom{a_i}{j} \left(s_{A+j\Delta_i} - c_A \frac{\binom{b_i}{j}}{\binom{a_i}{j}} \right) \mathbf{x}^{A+K+j\Delta_i} \in h_i J.$$

But since $h_i J$ is flat over $K[\varepsilon]$ (recall that the Borel group acts on $T_z \mathcal{H}$) this can only happen if all the coefficients are zero. For $j = 0$ this says $s_A = c_A$, and this holds for any $A \in \mathcal{F} \setminus \mathcal{F}'$. This gives (iii) since the h_i together with the diagonal matrices generate the Borel subgroup. If $A + \Delta_i \in \mathcal{F} \setminus \mathcal{F}'$ as well then $l \geq 1$. Setting the $j = 1$ coefficient to be zero gives us $c_{A+\Delta_i} = \frac{b_i}{a_i} c_A$. \square

This gives us the following description of the Borel-eigenvectors of the tangent space:

Theorem 7.2. *Let z be a Borel-fixed point on \mathcal{H} . Let m be the regularity of the sheaf of ideals \mathcal{I} defining the subscheme corresponding to z . Let I be the ideal given by*

$$I_d = \begin{cases} H^0 \mathcal{I}(d) & \text{if } d \geq m \\ 0 & \text{otherwise} \end{cases}$$

Set \mathcal{F} to be the set of (exponent vectors of) monomials of I of degree m (which is a filter of $\mathcal{P}(m, n)$) and \mathcal{R} the set of (exponent vectors of) the degree m standard monomials. Then the infinitesimal deformation

$$\left(\mathbf{x}^A + \varepsilon \sum_{B \in \mathcal{R}} c_{AB} \mathbf{x}^B \mid A \in \mathcal{F} \right) = (c_{AB})_{A \in \mathcal{F}, B \in \mathcal{R}}$$

is Borel-fixed as an element of the tangent space to Z if and only if

1. There exists $K \in \mathbf{Z}^{n+1}$ of degree 0 such that $c_{AB} \neq 0 \implies B - A = K$.

2. let

$$\mathcal{F}' = \mathcal{F} \setminus \{A \in \mathcal{F} \mid \exists B \in \mathcal{R}, c_{AB} \neq 0\}$$

and

$$\mathcal{F}'' = \mathcal{F} \cup \{B \in \mathcal{R} \mid \exists A \in \mathcal{F}, c_{AB} \neq 0\}$$

Then

(i) \mathcal{F}' and \mathcal{F}'' are filters;

(ii) For each $A = (a_0, \dots, a_n) \in \mathcal{F} \setminus \mathcal{F}'$, $B = A + K = (b_0, \dots, b_n)$, if $A + \Delta_i \in \mathcal{F} \setminus \mathcal{F}'$ then $c_{A+\Delta_i, B+\Delta_i} = \frac{b_i}{a_i} c_{AB}$.

The characterization of the Borel-fixed tangent vectors in 7.2 is rather dry and unenlightening, so let us illustrate with some examples. Take the Borel-fixed ideal $I = (x^3, x^2y, xy^2, y^3, x^2z) \subseteq K[x, y, z]$. We depict the monomials in degree 3 as in figure 5, with the monomials in I shaded.

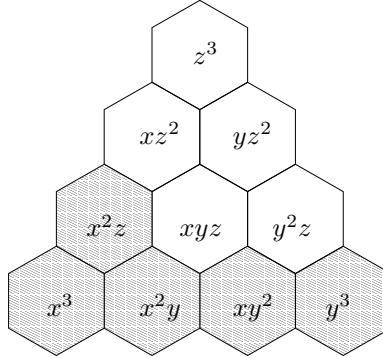


Figure 5: The monomials of degree 3.

We will represent a tangent vector (c_{AB}) by placing line segments on this picture, with a line segment extending from the hexagon representing A to that representing B if $c_{AB} \neq 0$, and labeling that line with the number c_{AB} if it is not 1. So for instance, on the left of figure 6 we depict the vector corresponding to the infinitesimally deformed ideal $(x^3, x^2y, xy^2, y^3 + \varepsilon xyz, x^2z) \subseteq K[x, y, z][\varepsilon]$, where $\varepsilon^2 = 0$. Condition 1 of theorem 7.2 is automatically satisfied in this example. Furthermore, here we have $\mathcal{F}' = \{x^3, x^2y, xy^2, x^2z\}$, and $\mathcal{F}'' = \{x^3, x^2y, xy^2, y^3, x^2z, xyz\}$, which are both filters of $\mathcal{P}(3, 2)$, and hence condition 2(i) is satisfied. Finally 2(ii) is immediate. Hence this vector is an eigenvector for the Borel subgroup. One can easily check the vector depicted on the right of figure 6 also is a Borel eigenvector.

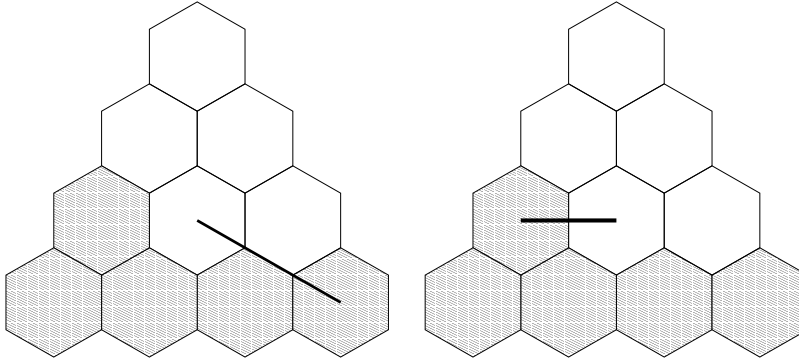


Figure 6: Two Borel eigenvectors.

Now consider instead the tangent vector depicted in figure 7. This represents the ideal $(x^3, x^2y, xy^2 + 2\varepsilon xyz, y^3 + 3\varepsilon y^2z, x^2z) \subseteq K[x, y, z][\varepsilon]$. Condition 1 of the theorem is satisfied with $K = (-1, 1, 0)$. In fact one can see that condition 1 just requires that all the line segments that appear must be rigid translates of each other. Condition 2(i) can be easily verified (note that a set of monomials in this picture form a filter if and only if they are closed under taking steps down and steps left). Finally condition 2(ii) holds: in the notation of the theorem we have $a_i = 3$ (the degree of the y variable in y^3) and $b_i = 2$ (the degree of the y variable in y^2z). So this vector is also a Borel eigenvector.

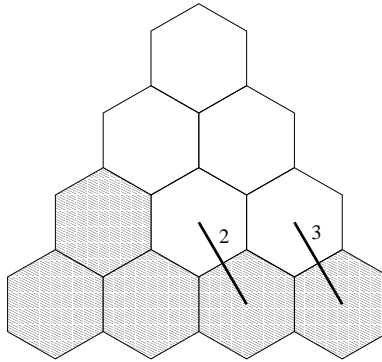


Figure 7: Another Borel eigenvector.

By examining possible Borel eigenvectors to be placed on figure 5 one can see that the three depicted in figures 6 and 7 are the only eigenvectors (up to scalar multiples of course).

We conclude this chapter by giving an alternative way of viewing theorem 7.2. Note that it gives a characterization of those lines through the origin of the

tangent space of a Borel-fixed point which are fixed by the action of the Borel subgroup of $\mathrm{GL}(n+1, K)$. However the lines through the origin in the tangent space to any point on any scheme can be viewed as points in the blowup of the scheme at that point. These are called *infinitely near points*. In this language theorem 7.2 characterizes *the infinitely near Borel-fixed points on the Hilbert scheme*.

8 An infinitesimal version of Galligo's theorem

In this section we prove the main result of this paper, namely that an infinitesimal version of Galligo's theorem [Gal79] holds. First let us recall Galligo's result. Recall that $\mathrm{GL}_{n+1}(K)$ acts on the set of ideals of the polynomial ring over K with $n+1$ variables.

Theorem 8.1 (Galligo). *Let $I \subseteq K[x_0, x_1, \dots, x_n]$ be any ideal and let $>$ be a monomial order with, say, $x_0 > x_1 > \dots > x_n$. Then there exists a Zariski open (and therefore dense) subset U of $\mathrm{GL}_{n+1}(K)$ such that for $g \in U$ the initial ideal $\mathrm{in}_{>}(gI)$ is constant over $g \in U$ and Borel-fixed (that is fixed by the action of the subgroup of upper-triangular matrices).*

In order to state our infinitesimal version we first need to switch from monomials orders $>$ to weight vectors. A *weight vector* w is an element of \mathbf{Z}^{n+1} which we use to partially order monomials by associating associating to each monomial a *weight* $w(\mathbf{x}^A) = w \cdot A$:

$$\mathbf{x}^A >_w \mathbf{x}^B \iff w \cdot A > w \cdot B.$$

Note that it makes perfect sense to consider weight vectors as any element of \mathbf{R}^{n+1} . However we will restrict our weight vectors to have integer coordinates.

If $>$ is a monomial order, and m is a positive integer, then we will say that a weight vector w *induces $>$ in degree m* if for any monomials $\mathbf{x}^A, \mathbf{x}^B$ of degree m we have $\mathbf{x}^A > \mathbf{x}^B \iff \mathbf{x}^A >_w \mathbf{x}^B$. If w induces $>$ in degree m then w induces $>$ in every degree $m' \leq m$. For any monomial order $>$ and degree m there is a weight vector (in fact many!) which induce $>$ in degree m . The weight vector w will be said to *distinguish monomials in degree m* if for any two distinct monomials $\mathbf{x}^A, \mathbf{x}^B$ of degree m we have either $\mathbf{x}^A >_w \mathbf{x}^B$ or $\mathbf{x}^A <_w \mathbf{x}^B$. If w distinguishes monomials in degree m then w distinguishes monomials in every degree $m' \leq m$.

Set $S = K[x_0, \dots, x_n]$ and let $I \subseteq S$ be an ideal. If w is a weight vector then for $t \neq 0$ we set $l = l(t)$ to be the diagonal matrix with diagonal entries $t^{-w_0}, \dots, t^{-w_n}$. We then define the new ideal $I(t)$ by $I(t) = l(t) \cdot I$. This is the result of replacing every occurrence of the variable x_i in I by $t^{-w_i} x_i$, for each $i = 0, \dots, n$. Then the set $\{I(t) \mid t \neq 0\}$ forms a one-parameter family of ideals which we can view as a curve on the Hilbert scheme. We call limit ideal the *initial ideal with respect to w* :

$$\mathrm{in}_w(I) := \lim_{t \rightarrow 0} I(t).$$

If w induces the monomial order $>$ to a large enough degree (the Gotzmann number is plenty large enough) then $\text{in}_w(I) = \text{in}_>(I)$.

The goal of this section is to prove the following infinitesimal version of Galligo's theorem: If $I \subseteq S$ is an ideal, and if w is a weight vector, with say $w_0 > w_1 > \dots > w_n$, which distinguishes monomials in a large enough degree, then for g in a dense open subset of $\text{GL}_{n+1}(K)$ the family of ideals $\{(gI)(t)\}$ (as defined above) has for limit $\text{in}_w(gI)$ a Borel-fixed ideal (Galligo's theorem), and the direction this family, viewed as lying on the Hilbert scheme, approaches this limit is itself Borel-fixed. We remark that since the Borel group fixes the limit point on the Hilbert scheme, it descends to an action on its tangent space.

Fix a projective space \mathbf{P}^n with homogeneous coordinate ring $S = K[x_0, \dots, x_n]$. For $r > 0$ and $m > 0$ consider the vector space $\wedge^r S_m$ of r -fold wedge products of degree m homogeneous forms in S . In this space we call the wedge product of r pair-wise distinct monomials $\mathbf{x}^{B_1} \wedge \dots \wedge \mathbf{x}^{B_r}$ a *state*. Two states $\wedge \mathbf{x}^{B_i} = \mathbf{x}^{B_1} \wedge \dots \wedge \mathbf{x}^{B_r}$ and $\wedge \mathbf{x}^{C_i} = \mathbf{x}^{C_1} \wedge \dots \wedge \mathbf{x}^{C_r}$ span the same linear (one dimensional) subspace iff there is a permutation $\sigma \in \text{Sym}(r)$ such that $B_i = C_{\sigma(i)}$, in which case $\wedge \mathbf{x}^{B_i} = (-1)^\sigma (\wedge \mathbf{x}^{C_i})$. Two such states will be called *equivalent*. The *associated monomial* of a state $\mathbf{x}^{B_1} \wedge \dots \wedge \mathbf{x}^{B_r}$ is the degree rm monomial $\mathbf{x}^{B_1} \dots \mathbf{x}^{B_r} = \mathbf{x}^{B_1 + \dots + B_r}$. If we are given a weight vector then we declare the *weight* of a state as the weight of its associated monomial. Equivalent states have the same weight. However nonequivalent states may still have the same weight; for instance $xz \wedge y^2$ and $xy \wedge yz$ each have weight will have the same weight for *any* weight vector as they both have the same associated monomial xy^2z .

Every element $f \in \wedge^r S_m$ can be written uniquely as a linear combination of states, if one ignores the distinction of equivalent weights. We define the *support* of f , denoted $\text{supp}(f)$ to be the set of those states appearing with non-zero coefficients. If we have a weight vector then for a given weight value N (which is an integer) we define $\text{supp}_N(f)$ to be the set of those states of weight N .

The individual summands in the expression of $f \in \wedge^r S_m$ as a linear combination of states will be referred to as the *terms* of f . Given a weight vector, the weight of such a term is just the weight of the associated monomial. For a given weight N we will write f_N for the sum of the terms of f with weight N . We will write $f_{\geq N}$ for $\sum_{M \geq N} f_M$, the sum of the terms of f with weight at least N . We analogously define $f_{>N}$, $f_{<N}$, and $f_{\leq N}$. We will also set $\text{supp}_{\geq N}(f) = \text{supp}(f_{\geq N})$, and similarly for $>, <, \leq$.

Given linearly independent homogeneous forms f_1, \dots, f_r of degree m , and a weight vector w that distinguishes monomials of degree m , we note that $f = f_1 \wedge \dots \wedge f_r$ has a unique term of maximal weight; namely pick f'_1, \dots, f'_r to span the same subspace as f_1, \dots, f_r , and such that the initial term (that is the term of largest weight) of each f'_i does not appear in any other f'_j . Then $f'_1 \wedge \dots \wedge f'_r$ differs from $f_1 \wedge \dots \wedge f_r$ only by a non-zero scalar multiple and it has a unique highest term of the form $c \cdot \text{in}_w(f'_1) \wedge \dots \wedge \text{in}_w(f'_r)$, with $c \neq 0$. We will write $\text{in}_w(f)$ for the corresponding state $\text{in}_w(f'_1) \wedge \dots \wedge \text{in}_w(f'_r) \in \text{supp}(f)$ of highest weight.

Let G be an $(n+1) \times (n+1)$ matrix with variable entries G_{ij} . If $f = f_1 \wedge \cdots \wedge f_r \in \wedge^r S_m$, we can express $Gf = (Gf_1) \wedge \cdots \wedge (Gf_r)$ as a linear combination of states whose coefficients are polynomials in the variables G_{ij} . Let $U(f) = U \subseteq \text{GL}(n+1, K)$ be the open set where none of the non-identically zero polynomials vanish. We make the following observations which follow immediately:

- (1) If $g, g' \in U = U(f)$ then $\text{supp}(gf) = \text{supp}(g'f)$;
- (2) If $g \in U$ and $g' \in \text{GL}_{n+1}$ is arbitrary then $\text{supp}(g'f) \subseteq \text{supp}(gf)$.
- (3) If $\lambda \in \text{GL}_{n+1}$ is a diagonal matrix then $\lambda U = U$

In addition we get the following:

Lemma 8.2. *Let $f = f_1 \wedge \cdots \wedge f_r \in \wedge^r S_m$, and let w be a weight vector which distinguishes monomials in degree rm . Set $l = l_w(t)$ be the diagonal matrix with diagonal entries $t^{-w_0}, \dots, t^{-w_n}$, for $t \neq 0$. Let $h \in \text{GL}(n+1, K)$ be an upper triangular matrix. If $g \in U$ then for any weight value N we have*

$$\text{supp}(h \cdot (lgf)_{\geq N}) = \text{supp}((lgf)_{\geq N})$$

for almost all values of t .

Proof. Note that the result of h on a state $\mathbf{x}^{C_1} \wedge \cdots \wedge \mathbf{x}^{C_r}$ is a linear combination of states of the form $\mathbf{x}^{D_1} \wedge \cdots \wedge \mathbf{x}^{D_r}$, where $w \cdot D_i \geq w \cdot C_i$. In particular every term other than $\mathbf{x}^{C_1} \wedge \cdots \wedge \mathbf{x}^{C_r}$ has weight strictly larger than $w \cdot (C_1 + \cdots + C_r)$.

Let $\tilde{f} = gf$. We compute

$$\begin{aligned} h\lambda\tilde{f} &= h\lambda \sum_M \tilde{f}_M \\ &= h \sum_M t^{-M} \tilde{f}_M \\ &= \sum_{M \geq N} t^{-M} h \cdot \tilde{f}_M + \sum_{M' < N} t^{-M'} h \cdot \tilde{f}_{M'} \\ &= [h \cdot (lgf)_{\geq N}] + [h \cdot (lgf)_{< N}] \end{aligned}$$

Now suppose that $h \cdot ((lgf)_{\geq N})$ contains a term $c(t)\mathbf{x}^{C_1} \wedge \cdots \wedge \mathbf{x}^{C_r}$ appearing with the coefficient $c(t) \neq 0$, which we consider as a Laurent polynomial in the variable t . Let $c'(t)$ be the coefficient of the same term in the right-hand sum. The degree of $c'(t)$ as a Laurent polynomial in t is strictly greater than $-N$, while that of $c(t)$ is at most $-N$. Hence the two cannot cancel as polynomials ($c(t) + c'(t) \neq 0$) and the state $\mathbf{x}^{C_1} \wedge \cdots \wedge \mathbf{x}^{C_r}$ lies in the support of $h \cdot (lgf)$ for infinitely many values of t (recall that the ground field K is infinite). By observations (2) and (3) above we see it lies in the support of lgf . This proves \subseteq .

Conversely, if $ct^{-M}\mathbf{x}^{C_1} \wedge \cdots \wedge \mathbf{x}^{C_r}$ is a term of $(lgf)_{\geq N}$ with weight $M = w \cdot (C_1 + \cdots + C_r) \geq N$, where $c \in K \setminus \{0\}$, then its coefficient in $h \cdot (lgf)_{\geq N}$ is a Laurent polynomial in t which has ct^{-N} as the only term with that power of t occurring. Hence it is not zero for infinitely many t . There are only finitely many terms, so we get the other containment, \supseteq . \square

Now we are in a position to prove an infinitesimal version of Galligo's Theorem. The idea is as follows. We take an ideal truncated in a large degree, say $I = (f_1, \dots, f_r)$. What we want is to deform the ideal I in generic coordinates to its initial ideal (given some monomial order), and show that as we get infinitesimally close to the initial ideal, we have something that is Borel-fixed. Deforming to the initial ideal is done by acting by a diagonal matrix with diagonal entries $t^{-w_0}, \dots, t^{-w_n}$, where $w = (w_0, \dots, w_n)$ induces our monomial order. Thus we will have a family of ideals parametrized by the variable t , and taking the limit as $t \rightarrow 0$ gives the initial ideal.

We see that as t gets small, the monomials largest in the term order begin to dominate (they have t coefficients with the smallest negative powers). The monomials next highest in the monomial order will then govern the *first order* behavior of the one parameter family. Specifically, if one takes the highest wedge product of the defining polynomials of a member of this family of ideals, then there is a unique term of highest weight (a fact we exploited to prove Galligo's theorem); however there may be many terms with the next highest weight, and it is these terms that will dominate to first order. The fact that there may be many terms of second highest weight presents a stumbling block. When we were just interested in the unique term with highest weight we could argue that after acting by an upper triangular matrix we could not have produced a new term with higher weight, since we had already picked the largest one possible. Now we need to control the terms with the second highest weight. However, though there may be many, 8.2 at least guarantees that the set of these states remains invariant.

The final problem we might encounter is that we don't really know what happens to the coefficients of the second highest weight terms after acting by an upper triangular matrix. To remedy this we will use lemma 7.1 which essentially says these coefficients are a red herring. Now on to the theorem.

Theorem 8.3. *Let z be a point on the Hilbert scheme corresponding to the subscheme $Z \subseteq \mathbf{P}^n$, and let m be the Gotzmann number for the Hilbert polynomial of Z . Let I be its defining ideal truncated at the degree m . Fix a weight vector $w = (w_0, \dots, w_n)$ which distinguishes monomials in degrees at least up to rm . As before set $l = l_w(t)$ to be the diagonal matrix with entries $t^{-w_0}, \dots, t^{-w_n}$. Let f_1, \dots, f_r be a basis for I (and thus a linear basis for I_m). Finally let $U = U(f)$, where $f = f_1 \wedge \dots \wedge f_r \in \wedge^r S_m$ (as defined above). Then for $g \in U$, the path on \mathcal{H} defined by the one-parameter family of ideals $\{l_w(t)gI\}$ has as limit as $t \rightarrow 0$ a Borel-fixed point, and the tangent vector to this path at that point is an eigenvector for the Borel group of upper triangular matrices.*

Proof. We have $gI = (gf_1, \dots, gf_r)$. Let $\tilde{f}_1, \dots, \tilde{f}_r$ be a new basis for gI where $\text{in}_w(\tilde{f}_i) = \mathbf{x}^{A_i}$ and this term appears in no other \tilde{f}_j , so that the initial ideal of gI is $(\mathbf{x}^{A_1}, \dots, \mathbf{x}^{A_r})$. We already know that this is Borel-fixed (Galligo's theorem). Let \mathcal{F} be the filter of exponent vectors $\{A_i\}$ and \mathcal{R} the order ideal of all other exponent vectors in degree m . Thus we can write

$$\tilde{f}_i = \mathbf{x}^{A_i} + \sum_{B \in \mathcal{R}} c_{A_i, B} \mathbf{x}^B$$

and therefore

$$\lambda \tilde{f}_i = t^{-w \cdot A_i} \mathbf{x}^{A_i} + \sum_{B \in \mathcal{R}} c_{A_i, B} t^{-w \cdot B} \mathbf{x}^B$$

For $t \neq 0$ then we find that λgI is generated by (f'_1, \dots, f'_r) , where for each i we set

$$f'_i := t^{w \cdot A_i} \lambda \tilde{f}_i = \mathbf{x}^{A_i} + \sum_{B \in \mathcal{R}} c_{A_i, B} t^{w \cdot (A_i - B)} \mathbf{x}^B. \quad (4)$$

Among the set of all differences $A_i - B$ with $c_{A_i, B} \neq 0$ choose one $K = A_i - B$ such that $w \cdot K$ is minimal. Let

$$\mathcal{F}' = \mathcal{F} \setminus \{A_i \in \mathcal{F} \mid B = A_i + K \in \mathcal{R} \text{ and } c_{A_i, B} \neq 0\}$$

and

$$\mathcal{F}'' = \mathcal{F} \cup \{B \in \mathcal{R} \mid A_i = B + K \in \mathcal{F} \text{ and } c_{A_i, B} \neq 0\}.$$

As $t \rightarrow 0$ the smallest powers of t dominate and we see the tangent vector is given by setting to zero all powers of t greater than $w \cdot K$. Thus the tangent vector (as an ideal in $S[\varepsilon]$) is given by the basis

$$\{\mathbf{x}^{A_i} \mid A_i \in \mathcal{F}'\} \cup \{\mathbf{x}^{A_i} + \varepsilon c_{A_i, A_i+K} \mathbf{x}^{A_i+K} \mid A_i \in \mathcal{F} \setminus \mathcal{F}'\}.$$

Note that this is an eigenvector for the maximal torus by 6.1, of type (\mathcal{F}', K) (see section 7 for the definition of *type*). By lemma 7.1 what we need to show is that after acting by an upper triangular matrix we get a vector with the same type. To do this return momentarily to the ideal $lgI = (f'_1, \dots, f'_r)$, for $t \neq 0$. From equation 4 we see that after expanding $f'_1 \wedge \dots \wedge f'_r$ we will have $\mathbf{x}^{A_1} \wedge \dots \wedge \mathbf{x}^{A_r}$ as the highest weight term, with weight $N = w \cdot (A_1 + \dots + A_r)$, and the second highest weight occurring is $N_1 = w \cdot (A + K)$. Specifically, if $\alpha = \mathbf{x}^{A_1} \wedge \dots \wedge \mathbf{x}^{A_r}$, and if for $A_i \in \mathcal{F} \setminus \mathcal{F}'$ we set $\alpha_i = \mathbf{x}^{A_1} \wedge \dots \wedge \mathbf{x}^{A_i+K} \wedge \dots \wedge \mathbf{x}^{A_r}$, (that is replace the monomial $\mathbf{x}^{A_i} \in \mathcal{F} \setminus \mathcal{F}'$ with \mathbf{x}^{A_i+K}), then we have

$$f'_1 \wedge \dots \wedge f'_r = \alpha + \sum_{A_i \in \mathcal{F} \setminus \mathcal{F}'} c_{A_i, A_i+K} t^{w \cdot K} \alpha_i + (\text{terms of lower weight}).$$

Lemma 8.2 gives us for any upper-triangular matrix h that

$$\text{supp}(h \cdot (\lambda g f)_{\geq N_1}) = \text{supp}(\lambda g f)_{\geq N_1}$$

for almost all values of t . Thus this holds for t in a Zariski open subset of $\mathbf{A}^1 \setminus 0$. Letting $t \rightarrow 0$ we see the tangent vector which has type (\mathcal{F}', K) still has type (\mathcal{F}', K) after acting by h . Since h was arbitrary, lemma 7.1 says this vector is an eigenvector for the Borel subgroup of upper-triangular matrices. \square

Some comments are in order. First we should note that the open subset U in theorem 8.3 is smaller than that used for Galligo's theorem. Thus "generic coordinates" has a stricter interpretation here. That said, we could have defined a larger open set on which the theorem still holds, but doing so drastically reduces a considerable degree of clarity.

Second we should comment on the choice of weight vector w . In theorem 8.3 we chose w to distinguish monomials up to the large degree rm , where m is the Gotzmann number, and $r = \dim I_m$. First off we could of chose the m simply as the degree of definition of the initial ideal. We simply chose to avoid over complicating the statement. Second we choose the large degree rm to ensure that states with distinct weights are weighted with distinct powers of the parametrizing variable t . However any weight vector inducing the term order only up to degree m already induces the term order, in the sense that the initial ideal with respect to the weight vector is the same as that with respect to the term order. Thus our condition on w is considerably more strict. Put another way, given an ideal in generic coordinates, we can define the *first order Gröbner fan* by taking the open chambers to be those weight vectors producing the same Borel eigenvector. This fan is finer than the typical Gröbner fan. Hence distinct weight vectors which induce the same term order may still produce different Borel eigenvectors. A weight vector that lies on a wall of the first order Gröbner fan, but in an open chamber of the typical Gröbner fan, will not give a Borel eigenvector.

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